Minimum LM Unit Root Test with Two Structural Breaks

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Abstract

The endogenous two-break unit root test of Lumsdaine and Papell (1997) is derived assuming no structural breaks under the null. Thus, rejection of the null does not necessarily imply rejection of a unit root per se, but may imply rejection of a unit root without break. Similarly, the alternative need not imply trend-stationarity with breaks, but may indicate a unit root with breaks. In this paper, we propose an endogenous two-break LM unit root test that allows for breaks under both the null and alternative hypotheses. An important feature of our test is that rejection of the null unambiguously implies trend-stationarity.

JEL classification: C12, C15, and C22

Key words: Lagrange Multiplier, Unit Root Test, Structural Break, Endogenous Break

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I. Introduction

Since the influential paper of Perron (1989), researchers have noted the importance of allowing for a structural break in unit root tests. Perron (1989) showed that the ability to reject a unit root decreases when the stationary alternative is true and an existing structural break is ignored. Perron (1989) used a modified Dickey-Fuller (hereafter DF) unit root test that includes dummy variables to account for one known, or "exogenous," structural break. Subsequent papers modified the test to allow for one unknown break point that is determined "endogenously" from the data. One widely used endogenous procedure is the "minimum test" of Zivot and Andrews (1992, hereafter ZA), which selects the break point where the *t*-statistic testing the null of a unit root is the most negative. Given a loss of power from ignoring one break, it is logical to expect a similar loss of power from ignoring two, or more, breaks in the one-break test. Lumsdaine and Papell (1997, hereafter LP) contribute in this direction and extend the minimum ZA unit root test to include two structural breaks.

One important issue common to the ZA and LP (and similar) endogenous break tests is that they assume no break(s) under the unit root null and derive their critical values accordingly. Thus, the alternative hypothesis would be "structural breaks are present," which includes the possibility of a unit root with break(s). As such, rejection of the null does not necessarily imply rejection of a unit root per se, but would imply rejection of *a unit root without breaks*. This outcome calls for a careful interpretation of test results in empirical work. In the presence of a break under the null, researchers might incorrectly conclude that rejection of the null indicates evidence of a trend-stationary time series with breaks, when in fact the series is difference-stationary with breaks. Despite this fact, numerous empirical papers that employ these endogenous break unit root tests conclude that rejection of the null is evidence of trend-stationarity.[1]

The hypotheses implied in the above endogenous break unit root tests differ from those in Perron's (1989) exogenous break unit root test, which allowed for the possibility of a break under both the null and alternative hypotheses. Allowing for breaks under the null is important in Perron's test; otherwise, the unit root test statistic will diverge as the size of a break under the null increases. A similar divergence occurs in the endogenous break unit root tests. Nunes, Newbold, and Kuan (1997) and Lee and Strazicich (2001) provide evidence that assuming no break under the null in endogenous break tests causes the test statistic to diverge and lead to significant rejections of the unit root null when the data generating process (DGP) is a unit root with break(s).[2]

As a remedy to the limitations of the LP test noted above, we propose a two-break minimum LM unit root test in which the alternative hypothesis unambiguously implies trend-stationarity. Our testing methodology is based on the Lagrange Multiplier (LM) unit root test as initially suggested in Schmidt and Phillips (1992, hereafter SP). Whereas assuming no break(s) under the null might be necessary in the LP test to make the test statistic invariant to break point nuisance parameters, this assumption is not required in the LM test as the distribution is invariant to break point nuisance parameters.[3]

Our paper proceeds as follows. Section II discusses the asymptotic properties of the endogenous two-break LM unit root test. Section III examines the test performance in simulations. Section IV examines the Nelson and Plosser (1982) data and compares results to the LP test. Section V summarizes and concludes. Throughout the paper, the symbol "® " denotes weak convergence of the associated probability measure.

II. Test Statistics and Structural Breaks under the Null

Perron (1989) considered three structural break models as follows. The "crash" Model A allows for a one-time change in level; the "changing growth" Model B allows for a change in trend slope; and Model C, which allows for a change in both the level and trend. Consider the DGP as follows:

$$y_t = d\,'Z_t + e_t\ ,\quad e_t = b\ e_{t-1} + e_t\ , \tag{1}$$

where Z_t is a vector of exogenous variables and e_t ~ iid $N(0, s^2)$.[4] Two structural breaks can be considered as follows.[5] Model A allows for two shifts in level and is described by $Z_t = [1,\ t,\ D_{1t},\ D_{2t}]'$, where $D_{jt} = 1$ for t ³ $T_{Bj} + 1,\ j=1,2$, and zero otherwise. T_{Bj} denotes the time period when a break occurs. Model C includes two changes in level and trend and is described by $Z_t = [1,\ t,\ D_{1t},\ D_{2t},\ DT_{1t},\ DT_{2t}]'$, where $DT_{jt} = t$ for t ³ $T_{Bj} + 1,\ j=1,2$, and zero otherwise. Note that the DGP includes breaks under the null (b = 1) and alternative (b < 1) hypothesis in a consistent manner. For instance, in Model A (a similar argument can be applied to Model C), depending on the value of b, we have:

Null	$y_t = \mu_0 + d_1 B_{1t} + d_2 B_{2t} + y_{t-1} + v_{1t}$	(2a)
Alternative	$y_t = \mu_1 + g \cdot t + d_1 D_{1t} + d_2 D_{2t} + v_{2t}\ ,$	(2b)

where v_{1t} and v_{2t} are stationary error terms, $B_{jt} = 1$ for $t = T_{Bj} + 1,\ j=1,2$, and zero otherwise, and $d = (d_1,\ d_2)'$. In Model C, D_{jt} terms are added to (2a) and DT_{jt} terms to (2b), respectively. Note that the null model (2a) includes dummy variables B_{jt}. Perron (1989, p. 1393) showed that including B_{jt} is necessary to insure that the asymptotic distribution of the test statistic is invariant to the size of breaks (δ) under the null.[6] In the LP test it is assumed that $d_1 = d_2 = 0$ under the unit root null (thus, omitting B_{jt} terms; LP, p. 212) and critical values of the test were derived under this assumption. As previously noted, this assumption is required; otherwise, the distribution of the LP test will depend on break point nuisance parameters describing the location and magnitude of breaks under the null.

The two-break LM unit root test statistic can be estimated by regression according to the LM (score) principle as follows:

$$\Delta y_t = \delta '\Delta Z_t + \phi \tilde{S}_{t-1} + u_t \,, \tag{3}$$

where $\tilde{S}_t = y_t - \tilde{\psi}_x - Z_t\tilde{\delta}$, $t=2,..,T$, $\tilde{\delta}$ are coefficients in the regression of Δy_t on ΔZ_t, $\tilde{\psi}_x$ is given by $y_1 - Z_1\tilde{\delta}$ (see SP), and y_1 and Z_1 denote the first observations of y_t and Z_t, respectively. The unit root null hypothesis is described by $\phi = 0$ and the LM test statistics are given by:

$$\tilde{\rho} = T\cdot\tilde{\phi} \tag{4a}$$

$$\tilde{\tau} = t\text{-statistic testing the null hypothesis } \phi = 0 \,. \tag{4b}$$

Assuming that the innovations ε_t satisfy the regularity conditions of Phillips and Perron (1988, p. 336), we define two error variances that are assumed positive and to exist as follows:

$$\sigma_\varepsilon^2 = \lim_{T\to\infty} T^{-1} E(\varepsilon_1^2 + \ldots + \varepsilon_T^2)$$

$$\sigma^2 = \lim_{T\to\infty} T^{-1} E(\varepsilon_1 + \ldots + \varepsilon_T)^2 \,.$$

We additionally assume (i) the data are generated according to (1), with $Z_t = [1, t, D_{1t}, D_{2t}]'$ for Model A and $Z_t = [1, t, D_{1t}, D_{2t}, DT_{1t}, DT_{2t}]'$ for Model C and *(ii)* $T_{Bj}/T \to \lambda_j$

as T ® ¥, where $l = (l_1,\ l_2)$¢. Then, from the asymptotic results demonstrated in the Appendix, we can show that under the null hypothesis:

$$\tilde{r} \text{ ® } -\frac{1}{2}\frac{s_e^{\,2}}{s^2}\,[\int_0^1 \underline{V}_B^{(m)}(r)^2 dr]^{-1} \qquad (5a)$$

$$\tilde{t} \text{ ® } -\frac{1}{2}\frac{s_e}{s}\,[\int_0^1 \underline{V}_B^{(m)}(r)^2 dr]^{-1/2}\ , \qquad (5b)$$

where $\underline{V}_B^{(m)}(r)$ is defined for m = A and C, respectively.

An important implication of (5a) and (5b) is the invariance property. In the Appendix, we show that the expression $\underline{V}_B^{(A)}(r)$ is the same as a demeaned Brownian bridge, $\underline{V}(r) = V(r) - \int_0^1 V(r)dr$. This result implies that the asymptotic null distribution of the two-break LM unit root test for Model A is invariant to the location (l) and magnitude (d) of structural breaks. This property follows from the results shown in Amsler and Lee (1995) for their exogenous one-break LM unit root test. Fortunately, this outcome carries over to the endogenous break LM unit root test. As such, the asymptotic distribution of the endogenous break LM unit root test will not diverge in the presence of breaks under the null and is robust to their misspecification. Unfortunately, this invariance property does not strictly hold for Model C, as the asymptotic null distribution of the endogenous break LM test depends on l (see Appendix). However, unlike the LP test, the minimum LM unit root test statistic for Model C does not diverge in the presence of breaks under the null, even when the breaks are large (see Section III).

The two-break minimum LM unit root test determines the break points (T_{Bj}) endogenously by utilizing a grid search as follows:

$$LM_r = \underset{l}{Inf}\ \tilde{r}(l) \qquad (6a)$$

$$LM_{\tau} = \underset{\lambda}{Inf}\ \tilde{\tau}(\lambda) \ . \qquad (6b)$$

The break point estimation scheme is similar to that in the LP test; the break points are determined to be where the test statistic is minimized. As is typical in endogenous break tests, trimming of the infimum over $[\kappa, 1\text{-}\kappa]$ for some κ, say 10%, is utilized to eliminate end points. Then, utilizing the limit theory on continuity of the composite functional in Zivot and Andrews (1992), the asymptotic distribution of the endogenous two-break LM unit root tests can be described as follows:

$$LM_{\rho} \rightarrow \underset{\lambda}{Inf}\ [-\frac{1}{2}\frac{\sigma_{\varepsilon}^{2}}{\sigma^{2}}(\int_0^1 \underline{V}_B^{(m)}(r)^2 dr)^{-1}\ ] \qquad (7a)$$

$$LM_{\tau} \rightarrow \underset{\lambda}{Inf}\ [-\frac{1}{2}\frac{\sigma_{\varepsilon}}{\sigma}(\int_0^1 \underline{V}_B^{(m)}(r)^2 dr)^{-1/2}] \ . \qquad (7b)$$

Critical values are derived using 50,000 replications for the exogenous break tests and 20,000 replications for the endogenous break tests in samples of T = 100.[7] Pseudo-iid $N(0,1)$ random numbers are generated using the Gauss (version 3.2.12) RNDNS procedure.[8] Results are shown in Table 1 and 2.

III. Simulations

This section examines simulation experiments to evaluate performance of the two-break minimum LM unit root test. Since performance of the LM_{ρ} test statistic is similar, we discuss only LM_{τ}. To highlight the invariance results, we first examine an exogenous version of the two-break LM test and then proceed to the endogenous test. Simulations are performed using 20,000 replications in the exogenous test and 5,000 replications in the endogenous test in samples of T = 100. Throughout, R denotes the number of structural breaks, λ is a vector denoting location of the breaks, and δ is a vector denoting the magnitude of breaks in the DGP. R_e and λ_e denote the values assumed in the test regression. All measures of size and power are reported using 5% critical values.

A. Exogenous Break Test

Simulation results using the exogenous two-break LM unit root test are reported in Table 3. We first examine Model A (two level shifts). Experiment 1 investigates the effects of assuming two breaks when no breaks are present. The results show no significant size distortion, implying that it does not hurt to allow for breaks when they do not exist. Note that the power of the LM test under the alternative (β = 0.9) in this baseline case is higher than that of the LP test (reported in parenthesis). As such, these findings are similar to those noted by Stock (1994) when comparing power of the no-break LM unit root test with no-break DF tests.

Experiment 2 investigates invariance properties using breaks of different locations (λ) and size (d). These findings clearly demonstrate the invariance properties of the LM test. Regardless of the location and magnitude of breaks under the null, the two-break LM unit root test rejects the null at 4.8%. As expected, under the null with break, the LP test exhibits over-rejections, which increase with the magnitude of the breaks. As previously noted, the greater rejections of the null in the LP test can be viewed as demonstrating high power when the alternative hypothesis is "structural breaks are present," or as spurious rejections when the null includes a unit root with break.

Experiment 3 examines effects of under-specifying the number of breaks (R_e < R). As expected, the two-break LM test is mostly invariant under the null to assuming too few breaks. Under the alternative there is a loss of power, which suggests that we should allow for breaks to increase power. Experiment 4 examines effects of incorrectly specifying the break points. Again, the two-break LM unit root test is mostly invariant to assuming incorrect break points under the null, while there is a loss of power under the alternative.

Results of the exogenous two-break LM unit root test for Model C (two level and trend shifts) are similar to those for Model A, except that the test statistic is no longer invariant to the location of breaks (l) under the null, but is nearly so. As with Model A, the LM test remains invariant to the size of breaks (*d*) under the null. Most important, the two-break LM test for Model C does not exhibit high rejections in the presence of breaks under the null. Experiment $3'$ and $4'$ show a negative size distortion when the number of breaks is under-specified or their location is incorrect.

B. Endogenous Break Test

Simulation results for the endogenous two-break LM unit root test are displayed in Table 4. We first examine the results for Model A (two level shifts). Experiment 5 compares 5% rejection rates using different break locations and magnitudes. Overall, the endogenous LM unit root test performs well in the presence of breaks under the null and shows no serious size distortions. In contrast, the endogenous two-break LP test exhibits significant rejections in the presence of breaks under the null, and more so as the magnitudes increase. In contrast to the LP test, the endogenous two-break LM test does not over-reject in the presence of breaks under the null. In addition, these results indicate that the same critical values can be utilized in the minimum LM test regardless of the location and size of breaks under the null. Under the alternative, we observe in

Experiment 6 that the power of the LM test is relatively stable for moderate size breaks. For relatively large breaks of $d = (10, 10)'$ a loss of power is observed. However, this result may not be surprising given that the time series would exhibit big swings and thus a low frequency would dominate the spectrum.

Simulation results for Model C are shown in the bottom of Table 4. The endogenous two-break LM unit root test has slightly greater size distortions than in Model A, but rejection rates are still close to 5%. Most important, as in Model A, the LM test does not diverge and remains free of the over-rejections observed in the LP test when breaks occur under the null. Thus, the endogenous two-break LM test may still be utilized for Model C, but for greater accuracy critical values should be employed corresponding to the break points (see Table 2).

As noted in Table 4, the two-break LP test exhibits over-rejections in the presence of breaks under the null, but seemingly high power under the alternative. Given the common interest in a trend-stationarity alternative, a more appropriate power comparison for Model C can be made by examining the *size-adjusted* power, which uses adjusted critical values corresponding to the magnitude of breaks. While the un-adjusted power of the LP test appears high, especially when the magnitude of breaks is large, the *size-adjusted power* is comparable to the endogenous LM test. In Experiment 6′ the *size-adjusted power* of the LP test is 0.096, 0.061, 0.059, 0.063, and 0.073, which is somewhat lower than that of the LM test.

Accuracy of estimating the break points is examined on the right side of Table 4. For Model A, the minimum LM test estimates break points reasonably well under the alternative, while the accuracy declines in Model C. In simulation results not reported here, we show that the LP test tends to select break points most frequently at T_{Bj}-1, which is not the case for the LM test.[9]

IV. Empirical Tests

In this section, the two-break minimum LM unit root test is applied to the Nelson and Plosser (1982) data. We use an augmented version to correct for serial correlation. Results are compared with the two-break minimum LP test. The Nelson and Plosser data comprise fourteen annual time series ranging from 1860 (or later) to 1970 and have the advantage of being extensively examined in the literature. All of the series are in logs except the interest rate. In each test, we determine the number of lagged augmentation terms by following the "general to specific" procedure described in Perron (1989) and suggested in Ng and Perron (1995). Starting from a maximum of k = 8 lagged terms, the procedure looks for significance of the last augmented term. We use the 10% asymptotic normal value of 1.645 on the t-statistic of the last first-differenced lagged term. After determining the "optimal k" at each combination of two break points, we determine the breaks to be where the endogenous two-break LM t-test statistic is at a minimum. To do so, we examine each possible combination of two break points over the time interval [.1T, .9T] (to eliminate end points). We follow Perron (1989) and ZA and assume Model A in all series except for the real wage and S & P 500 stock index, in which case Model C is assumed.

Overall, we find stronger rejections of the null using the LP test than with the LM test. At the 5% significance level, the null is rejected for six series with the LP test and four series with the LM test.[10] For example, while the null is rejected at the 5% significance level for Real GNP, Nominal GNP, Per-capita Real GNP, and Employment using the LP test, the null is rejected only at higher significance levels with the LM test.[11] As previously noted, the LP test often selects break points one period before the LM test.

To investigate the potential for over-rejections using the LP test, we estimate the size of breaks under the unit root null. If coefficients of the one-time dummy variables B_{jt} are significant, then we expect the LP test to reject the unit root null hypothesis more often. The null model in (2a) is estimated using the first differenced series as follows. Briefly, for each possible combination of T_{B1} and T_{B2} in the interval [.1T, .9T] (to eliminate endpoints), we again include k-augmented terms using the general to specific procedure. After determining the optimal k at each combination of two break points, the breaks are determined to be where the Schwarz Bayesian Criterion statistic is minimized. The estimated break coefficients are shown in standardized units, along with other results, in Table 5. Break terms under the null are found to be significant in most series; with (absolute) magnitudes ranging from near 2 to 8. These results suggest that even modest sized breaks under the null can potentially lead to different inference findings, or at least to different levels of significance.

V. Concluding Remarks

In many economic time series, allowing for one structural break may be too restrictive. This paper proposes a two-break minimum LM test, which endogenously determines the location of two-breaks in level and trend and tests the null of a unit root. Contrary to the endogenous two-break unit root test of Lumsdaine and Papell (1997), the endogenous two-break LM test does not diverge in the presence of breaks under the null. As such, using the two-break minimum LM test researchers will not conclude that a time series is trend-stationary with breaks when it is actually difference-stationary with breaks. In conclusion, the two-break minimum LM unit root test can be seen as a solution to a limitation of the two-break minimum LP test that includes the possibility of a unit root with break(s) in the alternative hypothesis. Using the two-break minimum LM unit root test, rejection of the null hypothesis unambiguously implies trend-stationarity.

Footnotes

1. See Raj and Slottje (1994), Ashworth, Evans, and Teriba (1999), Mehl (2000), and Ben-David, Lumsdaine, and Papell (2002), among others, for examples of papers that employ the ZA or LP endogenous break tests and conclude that rejection of the null indicates trend-stationarity.

2. An anonymous referee convincingly points out that the high rejection rates in the LP test can be viewed as high power. This point is valid if the desired alternative is the existence of breaks. Otherwise, a problem arises in practice. If the null is rejected, one may then need to examine the source of the rejection, as the alternative would include a unit root with break. In this case, the question of whether a time series is trend-stationary or difference-stationary would remain. We take the view that it is desirable to employ tests that allow for structural change in a unit root process as well as under the alternative. One may pose the question "can structural change coincide with a unit root process?" We answer this question in the affirmative. First, we note that Perron (1989) allowed for a break under the null in his initial unit root test. Second, there may be no compelling reason to believe that the persistence of structural change should be treated differently from the intrinsic persistence of a unit root process. For example, Phillips (1998) suggests that attempts to eliminate structural change by including dummy variables is itself support for the unit root hypothesis because such adjustments attach a unit weight and suggest a persistent shock. Third, our view is also consistent with Harvey, Leybourne, and Newbold (2001), who suggest that a structural break under the unit root null can be interpreted as a large permanent shock or outlier.

3. Strictly speaking, the endogenous break LM unit root test is invariant to break point nuisance parameters only for Model A (level shifts). The LM test for Model C (level and trend shifts) is not invariant to nuisance parameters, but is nearly so. However, in no case does the LM test diverge or exhibit any systematic pattern of over-rejections in the presence of breaks under the null (see footnote 9).

4. The baseline SP-LM test statistics are driven via a likelihood function that assumes $e_t \sim iid$ normal, but the *iid* assumption can be relaxed to correct for serial correlation. The test statistic can easily be extended to the case of autocorrelated errors by assuming that $A(L)e_t = B(L)u_t$, wherein $A(L)$ and $B(L)$ are finite order polynomials with $u_t \sim iid\ (0, \sigma_u^2)$ (see Ahn, 1993, and Lee and Schmidt, 1994). Further, following Phillips (1987) and Phillips and Perron (1988), we can assume the same regularity conditions that permit a degree of heterogeneity and serial correlation in e_t. Then, to correct for autocorrelated errors, lagged augmented terms $D\tilde{S}_{t-j}$, $j=1,..,k$, can be included in (3) as in the augmented DF test. Alternatively, a corrected test statistic using consistent estimates of the error variances can be employed as in the Phillips and Perron test.

5. Model B is omitted from further discussion, as it is commonly held that most economic time series can be adequately described by Model A or C.

6. In revisions to their structural break unit root tests, Perron (1993) and Perron and Vogelsang (1992) again include B_t terms in their testing regressions of the additive outlier (AO) model to be consistent under the null. They note that without B_t included, the test statistic diverges as the size of a break under the null increases. The same would be true for the innovative outlier (IO) model.

7. LP used 2,000 replications to obtain their endogenous break test critical values.

8. Copies of the Gauss computer codes utilized in this paper can be obtained at the web site http://www.bus.ucf.edu/lee/gauss.

9. The problem of estimating break points at $T_{Bj}-1$ occurs when B_{jt} terms are included in the test regression and may be avoided if these terms are omitted as in LP (p. 212, equation 1). When B_{jt} terms are omitted, the estimated break points tend to move from $T_{Bj}-1$ to T_{Bj}; thus seeming to solve the problem of incorrect estimation. However, with or without B_{jt} terms in the test regression, the two-break LP test statistic diverges and over-rejects in the presence of breaks under the null. Results are available upon request.

10. The empirical results in Table 5 use critical values from Table 2 (Model A) and Table 3 (Model C) in Lumsdaine and Papell (1997) for the two-break minimum LP test with B_{jt} in the testing regression. For comparison, the critical values used in the two-break minimum LM test in Table 5 were derived using the same sample size and trimming as in LP (T = 125 and 1%). The LM test critical values are -4.571, -3.937, and -3.564 for Model A and -6.281, -5.620, and -5.247 for Model C, at the 1%, 5%, and 10% significance levels, respectively.

11. For the real wage and money stock the opposite is the case.

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Table 1. Critical Values of the Exogenous Two-Break LM Unit Root Test (T=100)

Model A

	1%	5%	10%
$\tilde{t}$	-3.610	-3.047	-2.763
$\tilde{r}$	-23.13	-17.80	-14.87

Note: Owing to the invariance property of the LM test, critical values for Model A are the same as those in Schmidt and Phillips (1992).

Model C

$\tilde{t}$

	l_2		
l_1	.4	.6	.8
.2	-4.82, -4.19, -3.89	-4.92, -4.31, -4.00	-4.76, -4.19, -3.88
.4	-	-4.91, -4.33, -4.03	-4.87, -4.32, -4.03
.6	-	-	-4.84, -4.19, -3.89

$\tilde{r}$

	λ_2		
λ_1	.4	.6	.8
.2	-38.1, -30.2, -26.4	-39.3, -31.6, -27.9	-37.2, -30.1, -26.3
.4	-	-39.1, -31.8, -28.1	-38.7, -31.7, -28.1
.6	-	-	-38.3, -30.2, -26.4

Note: Critical values are at the 1%, 5%, and 10% levels, respectively. λ_j denotes the location of breaks.

Table 2. Critical Values of the Endogenous Two -Break LM Unit Root Test
(T = 100)

Model A

	1%	5%	10%
LM_t	-4.545	-3.842	-3.504
LM_r	-35.726	-26.894	-22.892

Model C (I)

	1%	5%	10%
LM_t	-5.823	-5.286	-4.989
LM_r	-52.550	-45.531	-41.663

Note: In the DGP, λ_1 and λ_2 are assumed to be absent.

Model C (II)

LM_t

		l_2	
l_1	.4	.6	.8
.2	-6.16, -5.59, -5.27	-6.41, -5.74, -5.32	-6.33, -5.71, -5.33
.4	-	-6.45, -5.67, -5.31	-6.42, -5.65, -5.32
.6	-	-	-6.32, -5.73, -5.32

LM_r

		λ_2	
λ_1	.4	.6	.8
.2	-55.4, -47.9, -44.0	-58.6, -49.9, -44.4	-57.6, -49.6, -44.6
.4		-59.3, -49.0, -44.3	-58.8, -48.7, -44.5
.6			-57.4, -49.8, -44.4

Note: Critical values are at the 1%, 5%, and 10% levels, respectively. λ_j denotes the location of breaks.

Table 3. Rejection Rates of the Exogenous Two-Break LM_t Unit Root Test (*T = 100*)

Model A

Exp	DGP			Estimation		Under the Null (*b = 1.0)*	Under the Alternative (*b = 0.9)*
	R	*l¢*	*d¢*	R_e	l_e¢		
1	0	-	-	2	.25, .50	.048 (.040)	.248 (.114)
				2	.25, .75	.048 (.040)	.246 (.110)
				2	.50, .75	.049 (.039)	.247 (.105)
2	2	.25, .50	5, 5	2	.25, .50	.048 (.487)	.248 (.763)
			10, 10	2	.25, .50	.048 (.955)	.248 (.998)
	2	.25, .75	5, 5	2	.25, .75	.048 (.485)	.246 (.757)
			10, 10	2	.25, .75	.048 (.956)	.246 (.997)
3	2	.25, .50	5, 5	0	-	.055	.130
		.25, .50	5, 5	1	.25	.047	.152
		.25, .50	5, 5	1	.50	.046	.141
	2	.25, .50	10, 10	0	-	.039	.021
		.25, .50	10, 10	1	.25	.034	.041
		.25, .50	10, 10	1	.50	.033	.027
4	2	.25, .50	5, 5	2	.25, .75	.048	.149
			10, 10	2	.25, .75	.034	.039

Note: For comparison, the corresponding size and power of the LP test is shown in parentheses.

Model C

Exp	DGP			Estimation		Under the Null (*b = 1.0)*	Under the Alternative (*b = 0.9)*
	R	*l¢*	*d¢*	R_e	l_e¢		
1′	0	-	-	2	.25, .50	.051 (.050)	.113 (.101)
				2	.25, .75	.047 (.052)	.112 (.101)
				2	.50, .75	.050 (.055)	.117 (.105)
2′	2	.25, .50	5, 5	2	.25, .50	.051 (.625)	.115 (.773)
			10, 10	2	.25, .50	.051 (.986)	.115 (.998)
	2	.25, .75	5, 5	2	.25, .75	.047 (.627)	.112 (.773)
			10, 10	2	.25, .75	.047 (.986)	.112 (.999)
3′	2	.25, .50	5, 5	0	-	.000	.000
		.25, .50	5, 5	1	.25	.002	.003
		.25, .50	5, 5	1	.50	.004	.004
	2	.25, .50	10, 10	0	-	.000	.000
		.25, .50	10, 10	1	.25	.000	.000
		.25, .50	10, 10	1	.50	.000	.000
4′	2	.25, .50	5, 5	2	.25, .75	.015	.013
			10, 10	2	.25, .75	.000	.000

Note: For comparison, the corresponding size and power of the LP test is shown in parentheses.

Table 4. Rejection Rates of the Endogenous Two-Break LM_t Unit Root Test

Model A

Exp	*l¢*	*d¢*	5% Rej.	Frequency of Estimated Break Points in the Range			
				T_B-1	T_B	T_B± 10	T_B± 30
			Under the null (b = 1)				
5	-	0, 0	.058 (.046)	-	-	-	-
	.25, .5	5, 5	.069 (.192)	.000	.116	.240	.668
	.25, .5	10, 10	.037 (.748)	.002	.234	.450	.744
	.25,.75	5, 5	.066 (.170)	.000	.032	.130	.599
	.2, .3	5, 5	.058 (.260)	.000	.244	.396	.623
			Under the alternative (b = .9)				
6	-	0, 0	.282 (.098)	-	-	-	-
	.25, .5	5, 5	.200 (.318)	.000	.226	.396	.726
	.25, .5	10, 10	.049 (.954)	.000	.538	.740	.851
	.25,.75	5, 5	.230 (.298)	.004	.101	.237	.673
	.2, .3	5, 5	.148 (.336)	.000	.325	.496	.681

Note: For comparison, the corresponding size and power of the LP test is shown in parentheses.

Model C

			Under the null (b = 1)				
5′	-	0, 0	.052 (.052)	-	-	-	-
	.25, .5	5, 5	.031 (.272)	.006	.016	.452	.903
	.25, .5	10, 10	.024 (.882)	.002	.016	.731	.995
	.25,.75	5, 5	.032 (.262)	.004	.018	.539	.950
	.2, .3	5, 5	.066 (.146)	.002	.000	.142	.502
			Under the alternative (b = .9)				
6′	.-	0, 0	.113 (.098)	-	-	-	-
	.25, .5	5, 5	.084 (.346)	.006	.041	.529	.938
	.25, .5	10, 10	.060 (.968)	.000	.041	.750	1.00
	.25,.75	5, 5	.074 (.348)	.006	.026	.592	.976
	.2, .3	5, 5	.107 (.246)	.002	.002	.198	.556

Note: For comparison, the corresponding size and power of the LP test is shown in parentheses. The corresponding *size adjusted power* of the LP test in Experiment 6′ is .096, .061, .059, .063, and .073, respectively.

Table 5. Empirical Results

SERIES	Model	LP $\hat{k}$	LP $\hat{T}_B$	LP *Stat.*	LM_t $\hat{k}$	LM_t $\hat{T}_B$	LM_t *Stat.*	Null Model $\hat{d}_1$*, $\hat{d}_2$* [a, b]	Null Model $\hat{T}_B$
Real GNP	A	2	1928	-7.00*	7	1920	-3.62	3.09, -2.67	1921
			1937			1941		(2.97, -2.65)	1929
Nominal GNP	A	8	1919	-7.50*	8	1920	-3.65	-4.84, -3.461	1920
			1928			1948		(-4.80, -3.27)	1931
Per-capita real GNP	A	2	1928	-6.88*	7	1920	-3.68	3.07, -2.60	1921
			1939			1941		(2.94, -2.57)	1929
Industrial Production	A	8	1917	-7.67*	8	1920	-4.32*	-3.73, -4.38	1920
			1928			1930		(-3.63, -4.13)	1931
Employment	A	8	1928	-6.80*	7	1920	-3.91	-2.90, 2.51	1931
			1955			1945		(-2.73, 2.33)	1941
Unemployment Rate	A	7	1928	-6.31*	7	1926	-4.47*	-3.43, 1.97	1917
			1941			1942		(-3.38, 1.83)	1920
GNP Deflator	A	8	1916	-4.74	1	1919	-3.18	3.88, -8.49	1917
			1920			1922		(3.73, -7.14)	1921
CPI	A	2	1914	-4.03	4	1916	-3.92	-2.44, -7.78	1920
			1944			1941		(-7.13, -2.41)	1930
Nominal Wage	A	7	1914	-5.85	7	1921	-3.84	-3.75, -2.98	1920
			1929			1942		(-3.62, -2.89)	1931
Real Wage	C	4	1921	-6.27	8	1922	-6.24*	-3.10, -.57	1931
			1940			1939		(-2.54, -3.01)	1945
Money Stock	A	8	1929	-6.22	7	1927	-4.31*	-3.54, -3.63	1920
			1958			1931		(-3.50, -3.50)	1931
Velocity	A	1	1883	-4.62	1	1893	-2.52	2.33, -2.28	1941
			1953			1947		(2.32, -2.27)	1944
Interest Rates	A	2	1931	-1.74	3	1949	-1.58	2.67, -2.50	1917
			1957			1958		(2.64, -2.45)	1921
SP500	C	1	1925	-6.37	3	1925	-5.57	3.12, 3.35	1928
			1938			1941		(4.82, 2.54)	1932

Note: * denotes significant at 5%. a: Standardized coefficients ($\hat{d}_i^* = \hat{d}_i/\hat{\sigma}$) are reported. b: t-statistics for $d_i = 0$ are given in parentheses. Data is the same as in Nelson and Plosser (1982).

Appendix

The Appendix describes the asymptotic properties of the endogenous two-break LM unit root test for Model A (two level shifts) and C (two level and trend shifts). Consider the following regression imposing the restriction $\beta = 1$ in (1):

$$\Delta y_t = \Delta Z_t'\delta + u_t \quad , \qquad \textbf{(A.1)}$$

where $u_t = \varepsilon_t$ under the null, and Z_t allows for exogenous trend break functions in addition to a linear trend function considered in SP. We define $\tilde{u}_t$ as the residual from the above regression:

$$\tilde{u}_t = \Delta y_t - \Delta Z_t' \tilde{\delta} = \varepsilon_t - \Delta Z_t' (\tilde{\delta} - \delta) \ . \qquad \textbf{(A.2)}$$

Then, the expression $\tilde{S}_t$ in the testing regression (3) can be obtained as a partial sum process of $\tilde{u}_t$. Letting $S_t = \sum_{j=2}^{t} \varepsilon_j$ and $[rT]$ be the integer part of rT, $r \in [0,1]$, we obtain:

$$\frac{1}{\sqrt{T}} \tilde{S}_{[rT]} = \frac{1}{\sqrt{T}} S_{[rT]} - \frac{1}{\sqrt{T}} (Z_{[rT]} - Z_1)'(\tilde{\delta} - \delta) \ .$$

(A.3)

As we will see below, the asymptotic properties of the LM test statistics are determined by the weak limit of this partial sum of residual process. Specifically, from regression (3), we obtain:

$$\tilde{\phi} = (\tilde{S}_1' M_{\Delta Z} \tilde{S}_1)^{-1} (\tilde{S}_1' M_{\Delta Z} \Delta y) \quad , \qquad \textbf{(A.4)}$$

where $\tilde{S}_1 = (\tilde{S}_1,..,\tilde{S}_{T-1})'$, $\Delta Z = (\Delta Z_2,..,\Delta Z_T)'$, $\Delta y = (\Delta y_2,..,\Delta y_T)'$, and $M_{\Delta Z} = I - \Delta Z(\Delta Z'\Delta Z)^{-1}\Delta Z'$. Then, following SP, it can be shown that:

$$\frac{1}{T^2} \tilde{S}_1' M_{\Delta Z} \tilde{S}_1 \rightarrow \sigma^2 \int_0^1 \underline{V}_B^{(m)}(r)^2 dr \ , \qquad \textbf{(A.5)}$$

where $\underline{V}_B^{(m)}(r)$, m=A,C, is the projection of the process $V_B^{(m)}(r)$ on the orthogonal complement of the space spanned by the trend break function $dz(\lambda,r)$ as defined over the interval $r \in [0,1]$. That is,

$$\underline{V}_B^{(m)}(r) = V_B^{(m)}(r) - dz(\lambda, r)\tilde{d} \ , \ for\ m\text{=A,C} \ , \qquad \textbf{(A.6)}$$

with

$$\tilde{\delta} = \underset{\delta}{\operatorname{argmin}} \int_0^1 (V_B^{(m)}(r) - dz(\lambda, r)\delta)^2 dr \ .$$

Here, $V_B^{(m)}(r)$ is the weak limit of the partial sum residual process $\tilde{S}_{[rT]}$ in (A.3) and is defined differently depending on the first difference of the exogenous trend break functions viz. $dz(\lambda,r)$, which is defined differently for each break model. In this section, we wish to show the explicit expression for $V_B^{(m)}(r)$, m=A,C. As a special case of the usual SP test not allowing for breaks, $dz(\lambda,r)$ is simply a constant function, $dz(\lambda,r) = 1$, and $V_B^{(m)}(r)$ becomes a standard Brownian bridge $V(r) = W(r) - rW(1)$.

For Model A with two level shifts, we let $Z_t = (t, W_t')'$, where $W_t = (D_{1t},..,D_{mt})'$ and $\boldsymbol{\delta} = (\delta_1, \delta_2')'$. Amsler and Lee (1995) derive asymptotic distributions of the LM test statistics with one known or exogenous structural break. Here we consider a more general case with a finite number of, say, $m << T$ structural breaks. Then, the partial sum process in (A.3) can be written as:

$$T^{-1/2}\tilde{S}_{[rT]} = T^{-1/2}S_{[rT]} - T^{-1}([rT]-1)\, T^{1/2}(\tilde{\delta}_1-\delta_1) - T^{-1}(W_{[rT]}-W_1)'\, T^{1/2}(\tilde{\delta}_2-\delta_2). \quad \textbf{(A.7)}$$

The first term on the right-hand side of (A.7) follows $T^{-1/2}S_{[rT]} \to \sigma W(r)$. For the second term, $\sqrt{T}(\tilde{\delta}_1-\delta_1) = (\frac{1}{T}\boldsymbol{\iota}'M_{DW}\boldsymbol{\iota})^{-1}\frac{1}{\sqrt{T}}\boldsymbol{\iota}'M_{DW}\boldsymbol{\varepsilon}$, where $M_{DW} = I - DW(DW'DW)^{-1}DW'$.

Here, $\frac{1}{T}\boldsymbol{\iota}'M_{DW}\boldsymbol{\iota} \to 1$, since $\boldsymbol{\iota}'DW = \boldsymbol{\iota}_m'$ ($1\times m$ vector of ones) and $\boldsymbol{\iota}'M_{DW}\boldsymbol{\iota} = T-m$. Then,

$$\frac{1}{\sqrt{T}}\boldsymbol{\iota}'M_{DW}\varepsilon = \frac{1}{\sqrt{T}}\sum_{j=2}^{T}\varepsilon_j - \frac{1}{\sqrt{T}}\sum_{i=1}^{m}\varepsilon_{Tbi+1} \to \sigma W(1) \ ,$$

$$T^{-1}([rT]-1)\, T^{1/2}(\tilde{\delta}_1-\delta_1) \to \sigma\, rW(1) \ .$$

We can show that the third term vanishes asymptotically. Since $W_{[rT]}-W_1 \to \boldsymbol{\iota}_m$,

$$\sqrt{T}(\tilde{\delta}_2-\delta_2) = (\frac{1}{T}DW'M_1 DW)^{-1}\frac{1}{\sqrt{T}}DW'M_1\varepsilon = o_p(1) \ ,$$

where $DW'M_1DW = I_m - I_mT^{-1} \to I_m$ and $DW'M_1\varepsilon = (\varepsilon_{TB1+1},..,\varepsilon_{TBm+1})' - \boldsymbol{\iota}_m\bar{\varepsilon}$. Thus, combining results, the terms in (A.7) follow:

$$T^{-1/2}\tilde{S}_{[rT]} \to \sigma[W(r) - rW(1)] = \sigma\, V(r) \ , \quad \textbf{(A.8)}$$

where $V(r)$ is a Brownian bridge. Thus, the expression $V_B^{(A)}(r)$ can be expressed as $V(r)$. This is the same expression as obtained from the usual SP test ignoring a break (see the equation before (A3.1) in SP, 1992, p. 283). In addition, note that:

$$\underline{V}_B^{(A)}(r) = V_B^{(A)}(r) - \tilde{d}_1 - [b_1(\lambda,r),..,b_m(\lambda,r)]\,\tilde{d}_2 \ ,$$

where $b_j(\lambda,r) = 1$ if $r = \lambda_j$, $j=1,..,m$, and 0 otherwise. The last term is again asymptotically negligible as shown for (A.7). Thus, we can show that $\underline{V}_B^{(A)}(r)$ in (A.5) will be a demeaned Brownian bridge $\underline{V}_B^{(A)}(r) = \underline{V}(r)$, where $\underline{V}(r) = V(r) - \int_0^1 V(r)dr$. Then, (A.5) becomes:

$$\frac{1}{T^2}\tilde{S}_1' M_{DZ}\tilde{S}_1 \to \sigma^2 \int_0^1 \underline{V}(r)^2 dr \ . \qquad \textbf{(A.9)}$$

Following SP, we can similarly show that for the second term in (A.4):

$$\frac{1}{T}\tilde{S}_1' M_{DZ} Dy = \frac{1}{T}\tilde{S}_1' M_{DZ}\, e = \frac{1}{T}\tilde{S}_1' \underline{e} \ \to\ -\frac{1}{2}\sigma_e^2 \ , \qquad \textbf{(A.10)}$$

where $\underline{e} = M_{DZ}\, e$ and the result $\frac{1}{\sqrt{T}}\tilde{S}_{[rT]} = \frac{1}{\sqrt{T}}\sum_{j=2}^{t}\underline{e}_j$. Combining this result with (A.9) we obtain for Model A:

$$\tilde{\rho} = T\tilde{\phi} \ \to\ -\frac{1}{2}\frac{\sigma_e^2}{\sigma^2}\Big[\int_0^1 \underline{V}(r)^2 dr\Big]^{-1} \ , \qquad \textbf{(A.11)}$$

which is the same limiting distribution as the usual SP statistic not allowing for breaks. Accordingly, the limiting distribution of $\tilde{\tau}$ is obtained as in SP.

For Model C with two breaks in both level and trend, we let $Z_t = (t, W_t')'$ where $W_t = (D_{1t}, D_{2t}, DT_{1t}, DT_{2t})'$. We additionally define $dz(\lambda,r) = [b_1(\lambda,r), b_2(\lambda,r), d_1(\lambda,r), d_2(\lambda,r)]$, where $d_j(\lambda,r) = 1$ if $r > \lambda_j$, $j=1,2$, and 0 otherwise. We note that the first two terms denoting a one time break are asymptotically negligible, as we observed for Model A. Thus, without a loss of generality we can simplify the algebra by using $dz(\lambda,r) \cong [d_1(\lambda,r), d_2(\lambda,r)]$, $d = (d_1, d_2)'$ and $Z_t \cong (t, DT_{1t}, DT_{2t})'$. Then letting $D_T = diag[T, T, T]$, we have as in SP:

$$D_T^{1/2}(\tilde{d} - d) = [D_T^{-1/2} DZ'DZ\, D_T^{-1/2}]^{-1} D_T^{1/2}\, DZ' e$$
$$\to\ \sigma\, B^{-1}\int_0^1 dz(\lambda,r)\, dW(r) \ , \qquad \textbf{(A.12)}$$

where $B = \begin{pmatrix} 1 & 1-\lambda_1 & 1-\lambda_2 \\ 1-\lambda_1 & 1-\lambda_1 & 1-\lambda_2 \\ 1-\lambda_2 & 1-\lambda_2 & 1-\lambda_2 \end{pmatrix}$ and $W(r)$ is a standard Wiener process. Then, the partial sum process in (A.3) follows as:

$$\frac{1}{\sqrt{T}}\tilde{S}_{[rT]} = \frac{1}{\sqrt{T}}S_{[rT]} - \frac{1}{\sqrt{T}}(Z_{[rT]} - Z_1)'D_T^{-1/2}[D_T^{1/2}(\tilde{d} - d)]$$

$$\rightarrow \sigma\ [\,W(r) - z(\lambda,r)B^{-1}\int_0^1 dz(\lambda,r)\, dW(r)] \equiv \sigma\ V_B^{(C)}(\lambda,r)\ , \qquad \textbf{(A.13)}$$

where $z(\lambda,r) = (r,\ dt_1(\lambda,r),\ dt_2(\lambda,r))$, $dt_j(\lambda,r) = r$, if $r > \lambda_j$, $j=1,2$, and 0 otherwise; and we denote $V_B^{(C)}(\lambda,r)$ as a break Brownian bridge. The expression $\underline{V}_B^{(C)}(\lambda,r)$, which we denote as a de-breaked Brownian bridge, is accordingly defined from (A.6) when λ is added to this expression to note that $\underline{V}_B^{(C)}(\lambda,r)$ depends on λ. Then, from (A.13):

$$\frac{1}{T^2}\tilde{S}_1'M_{DZ}\tilde{S}_1 \rightarrow \sigma^2\int_0^1 \underline{V}_B^{(C)}(\lambda,r)^2 dr\ . \qquad \textbf{(A.14)}$$

The result in (A.10) continues to hold for Model C. When we use $\underline{e} = M_{DZ}\, e$ with $\Delta Z_t \cong (1, D_{1t}, D_{2t})'$ it follows that:

$$\frac{1}{T}\tilde{S}_1'M_{DZ}\Delta y \rightarrow -\frac{1}{2}\sigma_e^2\ . \qquad \textbf{(A.15)}$$

As such, the asymptotic distributions of the LM test statistics for Model C are given by (A.14) and (A.15).