SETUP COSTS AND THE HERFINDAHL PRINCIPLE

by

Tam Vu, Eric Iksoon Im and Ujjayant Chakravorty[1]

Abstract

Most exhaustible resource sites (e.g., mines, landfills) and their users (e.g., cities) are spatially distributed. Gaudet, Moreaux, and Salant (2001) show that in the presence of setup costs, social efficiency may require that a site that is partially drawn down be abandoned by every user temporarily before some user returns to it. This result violates the "least cost first" principle of Herfindahl (1967). This paper shows that the Herfindahl principle is no longer violated when the setup costs are defrayed by charging an optimal fee for each unit of resource used. Moreover, this method of payment increases social welfare.

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[1] Respectively, University of Hawaii at Manoa, University of Hawaii at Hilo and University of Central Florida, Orlando. Correspondence: Tam Vu, tamv@hawaii.edu, 808-956-8169.

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Most mineral deposits and other exhaustible resource sites (e.g., landfills) and their users (e.g., cities) are spatially distributed. Hotelling's (1931) fundamental theorem of resource economics specifies how a resource may be extracted over time, but does not deal with the problem of multiple locations of resource sites and their users observed in practice. Moreover, extraction from a resource site almost always involves significant setup costs, such as in preparing and developing an oil field before the crude-oil pumping can begin.

In an important paper that generalizes the Hotelling framework to multiple sites and cities, Gaudet, Moreaux, and Salant (2001), henceforth called GMS, show that in the presence of setup costs, social efficiency may require that a site that is partially drawn down be abandoned by every city temporarily before some city returns to it. This result violates the "least cost first" principle of Herfindahl (1967), that the sites be used over time in increasing order of the unit cost of extraction.[2]

In the GMS model, the setup costs for each site are *implicitly* assumed to be paid off in lump sum at the time when they are incurred. In this paper, we show that when setup costs are paid by charging an optimal fee per unit of resource extracted, these costs can

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[2] Herfindahl's result was developed for the one city case. With many cities but no setup costs, Herfindahl's insight continues to apply for each city, as shown by GMS. Each city will use resource sites in order of their unit extraction cost. However, GMS argue that this result is violated in the presence of setup costs when a city may "vacillate," that is, abandon a site, move to a lower cost site and then return to the original site after a time delay.

be internalized as part of the marginal costs of extraction. As a result, the Herfindahl principle is no longer violated. More importantly, this method of payment increases social welfare.

Section I briefly reviews the GMS model, and shows how the setup costs are defrayed. In section II we propose a model in which setup costs are paid *via* a fee per unit of resource extracted. In section III, we show that this alternative payment mechanism is welfare enhancing. Section IV concludes the paper.

I. Lump Sum Payment of Setup Costs

In the GMS model, there are *m* resource sites and *n* cities located on a plane. They can be thought of as stocks of resources and potential users, respectively. The marginal costs of extraction and transportation from site *i* to city k, c_{ik}, are assumed to be constant and different for each city. Let $q_{ik}(t)$ be the rate of consumption of resource *i* by city *k*. The utility derived by city *k* from consuming the resource depends on the total consumption from all sites and is denoted by $U_k(q_k(t))$ where $q_k(t) = \sum_i q_{ik}(t)$. It is assumed to be strictly increasing and concave, with $U_k(0) = 0$ and $U_k'(0)$ finite.

Let $F_i > 0$ be the setup cost and τ_i the setup point, i.e., the time when setup costs are incurred.[3] Define $Q_i^0 > 0$ as the initial stock, $Q_i(t)$ as the corresponding residual stock of resource *i* at time *t,* and $q_i(t) = \sum_k q_{ik}(t)$ as the extraction rate of resource *i*. Let *r* be the social rate of time preference. Then the social planner's optimization problem in the GMS model is:

$$(1) \qquad \underset{\{q_{ik}(t)\}|\, T,\tau_i}{Max} \int_0^T e^{-rt} \sum_{k=1}^{m} \left(U_k(q_k(t)) - \sum_{i=1}^{n} c_{ik} q_{ik}(t) \right) dt - \sum_{i=1}^{n} e^{-r\tau_i} F_i$$

subject to

$$(2) \qquad \dot{Q}_i(t) = -\sum_k q_{ik}(t) = -\, q_i(t) \text{ ; and}$$

$$(3) \qquad q_{ik}(t) \geq 0, \quad q_{ik}(t) = 0 \quad \forall t < \tau_i, \quad Q_i(0) = Q_i^0, \quad Q_i(T) \geq 0 \quad (i = 1,...,n; k = 1,...,m).$$

For notational simplicity, define $\pi_i = e^{-r\tau_i}$, $\psi = \int_0^T e^{-rt} dt = (1 - e^{-rT})/r$ and $\tilde{F}_i = \pi_i F_i$ where $\tilde{F}_i$ is just the setup cost in present value terms.

Then, we can rewrite the last term in (1) as

$$(4) \qquad \sum_{i=1}^{n} \tilde{F}_i = \sum_{i=1}^{n} \pi_i F_i = \int_0^T e^{-rt} \left(\sum_{i=1}^{n} \frac{\pi_i}{\psi} F_i \right) dt$$

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[3] $F_i = 0$ is a trivial case, hence not considered.

so that the optimization problem becomes

$$(5) \qquad \underset{\{q_{ik}(t)\}|\,T,\,\tau_i}{Max} \int_0^T e^{-rt}\left[\sum_{i=1}^{m}\left(U_k(q_k(t))-\sum_{i=1}^{n}c_{ik}q_{ik}(t)\right)-\sum_{i=1}^{n}\frac{\pi_i}{\psi}F_i\right]dt\,,$$

and the current value Hamiltonian for (5) is

$$(6) \qquad H=\sum_{k=1}^{m}\left(U_k(q_k(t))-\sum_{i=1}^{n}c_{ik}q_{ik}(t)\right)-\sum_{i=1}^{n}\frac{\pi_i}{\psi}F_i \quad .$$

Equation (5) suggests that the setup cost for resource *i* paid off at time $t=\tau_i$ is *equivalent* to a continuous stream of fixed payments $\pi_i F_i/\psi$ over the time interval $\left[0,T\right]$. Note that the setup costs payment in this GMS framework are independent of the rate of extraction of the resource.

II. Payment of Setup Costs through a Unit Extraction Fee

Let δ_i be the fee charged for extracting each unit of the resource *i*. Then the fee paid by city *k* for extracting resource *i* at time *t* is $\delta_i q_{ik}(t)$. The present value of these payments aggregated over all cities during the time interval $[0,T]$ must equal $\widetilde{F}_i$. That is, defining $J(q_i(t))=\int_0^T e^{-rt}q_i(t)dt$,

$$\text{(7)} \qquad \tilde{F}_i = \int_0^T e^{-rt}\delta_i\, q_i(t)dt = \delta_i J(q_i(t))\,.$$

Substituting (7) into (1), we get

$$\text{(8)} \qquad \underset{\{q_{ik}(t)\},\delta_i \mid T,\tau_i}{Max} \quad \int_0^T e^{-rt}\sum_{k=1}^{m}\left[U_k(q_k(t)) - \sum_{i=1}^{n}(c_{ik}+\delta_i+\lambda_i(t))q_{ik}(t)\right]dt$$

subject to (2) and (3) and the *n* constraints in (7). With the setup costs integrated into the marginal costs of extraction, $c_{ik}+\delta_i$, the corresponding Lagrangian becomes

$$\text{(9)} \qquad L = \sum_{k=1}^{m}\left(U_k(q_k(t)) - \sum_{i=1}^{n}(c_{ik}+\delta_i+\lambda_i(t))q_{ik}(t)\right) \;+\sum_{i=1}^{n}\theta_i\left(\tilde{F}_i - \delta_i J(q_i(t))\right)$$

subject to (2) and (3) where θ_i are Lagrangian multipliers for the constraints in (7). The first-order conditions with respect to the control variables $\delta_i(>0)$ and $q_{ik}(t)\,(\geq 0)$ can be written as:

$$\text{(10)} \qquad \frac{\partial L}{\partial \delta_i} = -q_i(t) - \theta_i J(q_i(t)) = 0\,;$$

$$\text{(11)} \qquad \frac{\partial L}{\partial q_{ik}} = \left[U_k'(q_k(t)) - (c_{ik}+\delta_i+\lambda_i(t))\right] - \theta_i\delta_i\psi \leq 0 \;\; (= if \;\; q_{ik}(t)>0)\,.$$

Substituting δ_i from (7) into (11),

(12) $$\left[U_k'(q_k(t))-(c_{ik}+\lambda_i(t))\right]-\frac{\tilde{F}_i}{J(q_i(t))}\left(1+\theta_i\psi\right)\leq 0 \quad (= if \;\; q_{ik}(t)>0).$$

Given the first-order conditions (10) and (12), we can state

PROPOSITION 1: *There exists a unique global maximum for* $q_{ik}(t)$ *at* $t\in\left[0,T\right]$ *and for* $\delta_i\in\left(0,\tilde{F}_i/J(q_i(t))\right]\;\forall\, t\in\left[0,T\right]$.

Proof: see Appendix A.

Determinants of the Optimal Unit Extraction Fee

We now determine the extraction fee for resource *i*. If $q_{ik}(t)>0$, (12) must hold with equality. Substituting θ_i from (10) into (12) and $J(q_i)$ from (7) into (12), we have

(13) $$\frac{\psi}{\tilde{F}_i}q_{ik}(t)\,\delta_i^2-\delta_i+U_k'(q_k(t))-(c_{ik}+\lambda_i(t)) \;=\; 0.$$

Taking the definite integral of (13) over the time interval $\left[0,T\right]$, then summing over k, we get a quadratic equation in δ_i given by

(14) $$A_i\delta_i^2+B_i\delta_i+C_i=0$$

where

$$A_i = \sum_{k=1}^{m} \int_0^T \frac{\psi}{\tilde{F}_i} q_{ik}(t)dt = \frac{\psi}{\tilde{F}_i}\int_0^T \left(\sum_{k=1}^{m} q_{ik}(t)\right)dt = \frac{\psi}{\tilde{F}_i}\int_0^T q_i dt = \frac{\psi}{\tilde{F}_i} Q_i^0\,; \tag{15}$$

$$B_i = -\sum_{k=1}^{m} \int_0^T dt = -\sum_{k=1}^{m} T = -mT \text{ ; and}$$

$$C_i = \sum_{k=1}^{m}\int_0^T \left[U'(q_k(t)) - \left(c_{ik} + \lambda_i(t)\right)\right]dt\,.$$

Solving (14) for δ_i, we get

$$\delta_i^* = \frac{-B_i \pm \sqrt{B_i^2 - 4A_iC_i}}{2A_i}\,. \tag{16}$$

By Proposition 1, there exists a unique optimum value of δ_i denoted δ_i^*. Thus the quadratic equation in (14) must have a repeated root, i.e., $\sqrt{B_i^2 - 4A_iC_i} = 0$. We get

$$\delta_i^* = -\frac{B_i}{2A_i} = \frac{m\tilde{F}_i T}{2\psi Q_i^0} = \left(\frac{r}{1-e^{-rT}}\right)\left(e^{-r\tau_i}F_i\right)\frac{mT}{2Q_i^0} = \frac{e^{-r\tau_i} rF_i mT}{2(1-e^{-rT})Q_i^0} > \quad 0\,, \tag{17}$$

the positive sign being as expected.

Equation (17) shows the determinants of the unit extraction fee for resource *i*. It is an increasing function of the setup cost (F_i), the time for resources to be exhausted (*T*), and the number of users (*m*). It is a decreasing function of the date at which the setup cost for resource *i* is incurred (τ_i), and the initial stock of resource *i* (Q_i^0). Each of these factors affects the optimal unit fee in a way consistent with our intuition.

III. Welfare Comparison of the Two Payment Mechanisms

We now compare the two alternative mechanisms for paying setup costs: one through a lump sum payment and the other through unit extraction fees. We show that the latter is superior to the former. To facilitate the proof, let the setup costs be paid through a combination of the lump sum payment and the unit extraction fee, i.e., $\tilde{F}_i = \tilde{Z}_i + \tilde{R}_i$ where $\tilde{Z}_i \equiv \pi_i Z_i$, $\tilde{R}_i \equiv \int_0^T e^{-rt} \delta_i q_i(t) dt$, $Z_i \in \left[0, F_i / \pi_i\right]$, and $\delta_i \in \left[0, \tilde{F}_i / J(q_i(t))\right]$. Here Z_i denotes the portion of setup costs for resource *i* that is paid in lump sum at the setup point τ_i, $\tilde{Z}_i$ is its present value, and $\tilde{R}_i$ is the present value of the portion paid through the unit extraction fee. Then the present value of the setup costs can be written as

$$(18) \qquad \tilde{F}_i = \pi_i Z_i + \int_0^T e^{-rt} \delta_i \, q_i(t) dt = \int_0^T e^{-rt} \frac{\pi_i}{\psi} Z_i dt + \int_0^T e^{-rt} \sum_{k=1}^{m} \delta_i q_{ik}(t) \, dt .$$

If the optimal value for Z_i is zero, then the payment of setup costs through unit extraction fee must be superior to the lump sum payment. Substituting (18) into (1), the optimization problem becomes

$$(19) \qquad \underset{\{q_{ik}(t)\}, \delta_i, \tilde{Z}_i | T, \tau_i}{Max} \int_0^T e^{-rt} \left(\sum_{k=1}^{m} \left[U_k(q_k(t)) - \sum_{i=1}^{n} (c_{ik} + \delta_i + \lambda_i(t)) q_{ik}(t) \right] - \sum_{i=1}^{n} \frac{\pi_i}{\psi} Z_i \right) dt$$

subject to

(20) $\dot{Q}_i(t) = -\sum_{k=1}^{m} q_{ik}(t)$;

(21) $q_{ik}(t) \geq 0 \;\; \forall t \in [0,T], \quad Q_i(0) = Q_i^0, \quad Q_i(T) \geq 0$;

(22) $Z_i \geq 0$; and

(23) $\tilde{F}_i = \pi_i Z_i + \delta_i J(q_i(t))$.

Let θ_i be multipliers for the constraints in (23). Then the Lagrangian is given by

(24)
$$L = \sum_{k=1}^{m} \left(U_k(q_k(t)) - \sum_{i=1}^{n} \left(c_{ik} + \delta_i + \lambda_i(t) \right) q_{ik}(t) \right) - \sum_{i=1}^{n} \frac{\pi_i}{\psi} Z_i + \sum_{i=1}^{n} \theta_i \left[\tilde{F}_i - \pi_i Z_i - \delta_i J(q_i(t)) \right]$$

The first-order conditions for non-negativity of δ_i, Z_i, and q_{ik} may be written as

(25) $\frac{\partial L}{\partial \delta_i} = -q_i(t) - \theta_i J(q_i(t)) \leq 0 \qquad (= if \;\; \delta_i > 0)$;

(26) $\frac{\partial L}{\partial Z_i} = -\frac{\pi_i}{\psi} - \theta_i \pi_i = -\frac{\pi_i}{\psi}\left(1 + \theta_i \psi\right) \leq 0 \qquad (= if \;\; Z_i > 0)$; and

(27) $\frac{\partial L}{\partial q_{ik}} = U_k'(q_k(t)) - (c_{ik} + \delta_i + \lambda_i(t)) - \psi \theta_i \delta_i \leq 0 \quad (= if \;\; q_{ik}(t) > 0)$.

We now state

PROPOSITION 2: *In paying off the setup costs for a resource site, the optimal fee per unit of resource extracted is superior to the lump sum payment at the setup point.*

Proof: It suffices to show that $Z_i^* = 0$, Z_i^* denoting the optimum value of $Z_i \in [0, F_i]$. We do it in two parts: when (i) $Z_i \in [0, F_i)$ and (ii) $Z_i = F_i$.

(i). For $Z_i \in [0, F_i)$, there is a non-zero portion of setup costs paid through a unit extraction fee. Thus $\delta_i > 0$ which implies that (25) holds with equality:

$$(29) \quad \theta_i = -\frac{q_i(t)}{J(q_i(t))}.$$

Substituting (29) into (26), the Kuhn-Tucker conditions for Z_i become

$$(30) \quad \frac{\partial L}{\partial Z_i} = \frac{\pi_i}{\psi}\left(\frac{\psi\, q_i(t) - J(q_i(t))}{J(q_i(t))}\right) \leq 0 \quad (= \; if \; Z_i > 0).$$

Using (30) for $q_i(t) \geq 0$ in conjunction with (27), we can show that $Z_i^* = 0$. When $q_i(t) = 0$, (30) yields

$$(31) \quad \frac{\partial L}{\partial Z_i} = -\frac{\pi_i}{\psi} < 0$$

which implies that $Z_i^* = 0$. When $q_i(t) > 0$, there must be a resource *i* for which $q_{ik}(t) > 0$ so that (27) holds for equality:

$$(32) \quad U_k'(q_k(t)) - (c_{ik} + \delta_i + \lambda_i(t)) - \psi\theta_i\delta_i = 0.$$

Substituting δ_i and θ_i, respectively, from (28) and (29) into (32), then differentiating with respect to $q_{ik}(t)$, we obtain (see Appendix D for details)

$$\text{(33)} \qquad \frac{\partial L}{\partial Z_i} = \frac{\pi_i}{\psi}\left(\frac{\psi\, q_k(t) - J(q_i(t))}{J(q_i(t))}\right) = \frac{\pi_i}{\psi}\left(\frac{U_k^{"}(q_k(t)}{2\pi(\tilde{F}_i - \tilde{Z}_i)}\left[J(q_i(t))\right]^2\right) < 0$$

for all $Z_i \in \left[0, F_i\right)$, which also implies that $Z_i^* = 0$.

(ii). When $Z_i = F_i > 0$, (30) holds with equality. Solving for $q_i(t)$, we get

$$\text{(34)} \qquad q_i(t) = \frac{J(q_i(t))}{\psi} = \frac{1}{\psi r}\left(q_i(0) - e^{-rT} q(T)\right) \equiv \mu_i > 0$$

which can not be the optimal path for $q_i(t)$,[4] hence $Z_i^* = 0$.

■

IV. Concluding Remarks

Since it is optimal to repay the entire setup costs through unit extraction fees, replacing δ_i with δ_i^* in the optimization problem (8) and dropping the constraints (7), which must be satisfied by the solution δ_i^*, we have:

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[4] When $Z_i = F_i > 0$, $\delta_i = 0$. Further, $q_i(t) > 0$ in (34) implies $q_{ik}(t) > 0$. Hence, (27) can be rewritten as $U_k^{'}(q_k(t)) = \lambda_i(t) + c_{ij}$ from which we note that $q_k(t)$, and therefore $q_i(t)$ is a decreasing function of time, contradicting $q_i(t) = \mu_i$ since μ_i is a constant.

$$(37) \qquad \underset{\{q_{ik}(t)\}|\,\delta^*,T,\tau_i}{Max} \int_0^T e^{-rt} \sum_{k=1}^{m} \left[U_k(q_k(t)) - \sum_{i=1}^{n} \left(c_{ik} + \delta_i^* + \lambda_i(t) \right) q_{ik}(t) \right] dt$$

subject to

$$\dot{Q}_i(t) = -\sum_{k=1}^{m} q_{ik}(t), \quad i = 1,2,\dots,n,$$

$$q_{ik}(t) \geq 0, \quad q_{ik}(t) = 0 \quad \forall t < \tau_i, \quad Q_i(0) = Q_i^0, \quad Q_i(T) \geq 0.$$

The corresponding Hamiltonian is no longer constrained:

$$(38) \qquad \mathrm{H}^* = \sum_{k=1}^{m} \left[U_k(q_k(t)) - \sum_{i=1}^{n} \left(w_{ik} + \lambda_i(t) \right) q_{ik}(t) \right]$$

where $w_{ik} = c_{ik} + \delta_i^*$. It demonstrates that the setup costs of the resource sites have been completely internalized as part of the new marginal costs w_{ik}. The extraction sequence of the resource sites by each city must be strictly in the order of w_{ik}, not c_{ik}. The Herfindahl rule is preserved for each city: a city will never abandon a site and return to it after a delay, as suggested by GMS. The economic intuition is simple: in the long run, the social planner can increase social welfare by internalizing all costs, including setup costs, as variable.

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Appendix A: PROOF OF PROPOSITION 1

To prove that a unique global maximum exists for $q_{ik}(t)$ at $t \in [0,T]$ and for $\delta_i \in \left(0, \tilde{F}_i / J(q_i(t))\right]$ $\forall t \in [0,T]$, we only need to show that the Hessian for (9) is negative definite:

The second derivatives of L with respect to the control variables are:

(A1) $$L_{\delta_i \delta_i} = \theta_i \frac{\tilde{F}_i}{\delta_i^2} < 0,$$

(A2) $$L_{q_{ik} q_{ik}} = U_k''(q_k(t)) + \frac{\psi \tilde{F}_i}{\left[J(q_i(t))\right]^2}(1+\theta_i \psi) = \frac{1}{2} U''(q_k(t)) < 0,$$ and

(A3) $$L_{\delta_i q_{ik}} = -(1+\theta_i \psi) < 0$$

where

(A4) $$\theta_i = -\frac{q_i(t)}{J(q_i(t))} < 0$$ (from (10) for $\delta_i > 0$),

(A5) $$U_k''(q_k(t)) < 0$$ (by assumption), and

(A6) $$1+\theta_i \psi = -\frac{U''(q_k(t))}{2\psi \tilde{F}_i}\left[J(q_i(t))\right]^2 > 0.$$ (see Appendix B for proof).

Hence, the Lagrangian Hessian is negative definite:

(A7) $$H = \begin{bmatrix} L_{\delta\delta} & L_{\delta q_{ik}} \\ L_{q_{ik}\delta} & L_{q_{ik} q_{ik}} \end{bmatrix} < 0,$$

since

(A8) $$|H_1| = L_{\delta\delta} = \theta_i \frac{\tilde{F}_i}{\delta_i^2} < 0,$$ and

(A9) $$|H_2| = L_{\delta\delta} L_{q_{ik} q_{ik}} - \left(L_{q_{ik}\delta}\right)^2$$

$$= -\theta_i \psi (1+\theta_i \psi) - (1+\theta_i \psi)^2$$

$$= -\,(1+\theta_i\psi)\,(2\theta_i\psi+1) \quad > \quad 0\,,$$

because

(A10) $\quad (1+\theta_i\psi) > 0$ from (A6), and $(2\theta_i\psi+1) < 0$ for $\forall\ q_i(t)$ (See Appendix C).

Appendix B: PROOF OF (A6)

From (12) for $q_k(t) > 0$,

(B1) $$U_k^{'}(q_k(t)) - (c_{ik} + \lambda_i(t)) - \frac{\tilde{F}_i}{J(q_i(t))}\left(1 - \frac{q_i(t)}{J(q_i(t))}\psi\right) = 0\,.$$

Differentiating (B1) with respect to $q_k(t)$, we get

(B2) $$U_k^{"}(q_k(t)) + \frac{2\psi\tilde{F}_i}{\left[J(q_i(t))\right]^2}\left(\frac{J(q_i(t)) - q_i(t)\psi}{J(q_i(t))}\right) = 0\,.$$

Substituting q_i from (A4) into (B2), we obtain

(B3) $$1+\theta_i\psi = -\frac{U_k^{"}(q_k(t))}{2\psi\tilde{F}_i}\left[J(q_i(t))\right]^2\,.$$

Appendix C: PROOF OF (A10)

Let $\eta_i = 2\theta_i\psi + 1$. From (29), we have

(C1) $$\eta_i = \frac{-2q_i(t)\psi}{J(q_i(t))} + 1\,.$$

Differentiating (C1) with respect to $q_i(t)$, we get

(C2) $$\frac{\partial\eta_i}{\partial q_i(t)} = \frac{-2\psi}{J(q_i(t))}(1+\theta_i\psi) \quad < \quad 0\,.$$

That is, η_i is a monotonically decreasing function of $q_i(t) \geq 0$. Therefore, it suffices to show that $\eta_i < 0$ when $q_i(t) = 0$ to prove that $\eta_i < 0$ for all $q_i(t) \geq 0$. When $q_i(t) = 0$, both the numerator and the denominator in the first term on the right-hand side of (C1) go to zero. Therefore, using $L'Hopital's$ rule,

(C3) $$\eta_i\Big|_{q_i(t)=0} = \frac{\left.\frac{\partial}{\partial q_i(t)}\left(-2q_i(t)\psi\right)\right|_{q_i(t)=0}}{\left.\frac{\partial}{\partial q_i(t)}\left[J(q_i(t))\right]\right|_{q_i(t)=0}} + 1 = \frac{-2\psi}{\psi} + 1 = -1 \quad < \quad 0 .$$

Appendix D: PROOF OF (33)

Substituting δ_i from (28) and θ_i from (29) into (32), we have

(D1) $$U_k^{'}(q_k(t)) - (c_{ij} + \lambda_i(t)) - \frac{\tilde{F}_i - \tilde{Z}_i}{J(q_i(t))}\left(1 - \frac{q_i(t)}{J(q_i(t))}\psi\right) = 0 .$$

Differentiating (D1) with respect to $q_{ik}(t)$ yields

(D2) $$U_k^{"}(q_k(t)) + \frac{2\psi(\tilde{F}_i - \tilde{Z}_i)}{\left[J(q_i(t))\right]^2}\left(\frac{J(q_i(t)) - q_i(t)\psi}{J(q_i(t))}\right) = 0$$

which implies that

(D3) $$\frac{q_i(t)\psi - J(q_i(t))}{J(q_i(t))} = \frac{U_k^{"}(q_k(t))}{2\psi(\tilde{F}_i - \tilde{Z}_i)}\left[J(q_i(t))\right]^2 \quad < \quad 0$$

where $\tilde{F}_i - \tilde{Z}_i = \pi_i(F_i - Z_i) > 0$ since $Z_i \in \left[0, F_i\right)$ is assumed for (33).