Chapter 6
Matrix Algebra

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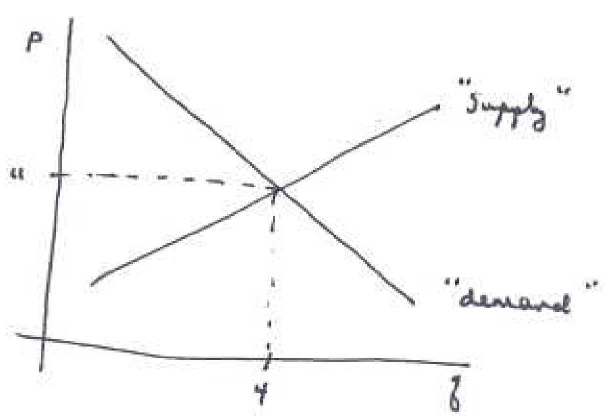
1. Suppose that we want to solve the following two equations

$p = 15 - q$ "demand"
$p = 3 + 2q$ "Supply"

From previous work, we know how to do this: Set

$15 - q = 3 + 2q$

OR, $3q = 12$; $q = 4$, $p = 11$



So, the solution gives equilibrium price and quantity in the market

2. Suppose now that we want to solve the three equation system

$$2x + y + z = 3$$
$$-x + 2y + 2z = 1$$
$$x - y - 3z = -6$$

Can solve the third equation: $x = y + 3z - 6$
Substitute solution into first two equations and solve these for y, z. Then determine x. We get: $x = 1$, $y = -2$, $z = 3$
(This is worked out on chapter 3 slides).

3. Suppose we wish to solve larger systems of equations, such as a 5 equation system. This would be quite tedious without a more efficient way to represent the equations to be solved.

4. Need to know matrix algebra

5. Basics

a. A matrix is a set of <u>elements</u> organized into rows and columns. Thus,

$$A = \begin{array}{cccc} & & & \\ \left[\begin{array}{ccc} 10 & 12 & 16 \\ 5 & 9 & 7 \end{array}\right] & \begin{array}{c} r1 \\ r2 \end{array} \\ \begin{array}{ccc} c1 & c2 & c3 \end{array} & \end{array}$$

"2x3 matrix"

$$B = \begin{bmatrix} 1 & 6 & -2 \\ 5 & 1 & -4 \\ -3 & 5 & 0 \end{bmatrix}$$

"3x3 matrix"

$$C = \begin{bmatrix} 1 & 2 \\ -3 & 4 \\ 5 & 6 \\ 7 & -8 \end{bmatrix}$$

"4x2 matrix"

In general

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

"m x n matrix"

b. Note that a matrix can be just one row or one column (referred to as a vector

$$\begin{bmatrix} 3 \\ 8 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 & -12 \end{bmatrix}$$

b. Equality of matrices

matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if and only if they have the same size (the same number of rows and columns) and $a_{ij} = b_{ij}$ for each i and j. This means that the corresponding elements in the two matrices are equal

Example $A = B$ if

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 4 & -2 \end{bmatrix}$$

c. Transpose of a matrix

The transpose of a matrix $A = [a_{ij}]$, denoted A^T is formed by interchanging the i^{th} row with the i^{th} column.

Example:

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

One obvious property of the transpose is that $(A^T)^T = A$

D. Special matrices

1) $$O = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix}$$ "zero matrix"

2) "Identity matrix" plays the role of the number 1 in scalar algebra

$$I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \text{; example } \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3) Diagonal matrix

$$D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

has ~~non~~ zero elements on the "off-diagonal" and non-zero elements on the diagonal

6. Homework: pp. 254-255
   #1, 19,

7. Matrix Addition and Scalar multiplication

a. Addition

1) Definition: If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $m \times n$ matrices then $A + B$ is the $m \times n$ matrix obtained by adding corresponding elements; i.e. $A + B = [a_{ij} + b_{ij}]$

2) Example:

$$A = \begin{bmatrix} 3 & 0 & -2 \\ 2 & -1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & -3 & 6 \\ 1 & 2 & -5 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 8 & -3 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

3) If A and B have different numbers of rows and columns then addition is not possible. Example:

$$A = \begin{bmatrix} 4 & 8 \\ 9 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 1 & 8 \\ 6 & 2 & 3 \\ 4 & 1 & 1 \end{bmatrix}$$

4) Properties of matrix addition

(a) $A + B = B + A$ "commutative property"

(b) $A + (B + C) = (A + B) + C$ "associative property"

(c) $A + O = A$ "identity property"

b. Scalar multiplication

1) Definition: If A is an $m \times n$ matrix and k is a scalar then $kA = [k a_{ij}]$

2) Example:

$$-3\begin{bmatrix} 1 & 0 & -2 \\ 2 & -1 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 6 \\ -6 & 3 & -12 \end{bmatrix}$$

3) Properties of scalar multiplication

a) $k(A+B) = kA + kB$

b) $(k_1 + k_2)A = k_1 A + k_2 A$

c) $k_1(k_2 A) = (k_1 k_2)A$

d) $0A = 0$

e) $k0 = 0$

c. matrix subtraction

1) Definition: If A and B are the same size, then by $A-B$, we mean that $A - B = A + (-B)$

2) Example:

$$A = \begin{bmatrix} 2 & 6 \\ -4 & 1 \\ 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 6 & -2 \\ 4 & 1 \\ 0 & 3 \end{bmatrix}$$

$$A - B = \begin{bmatrix} -4 & 8 \\ -8 & 0 \\ 3 & -1 \end{bmatrix}$$

8. Homework: pp. 261-262, #1, 5, 7, 17. Do a few more to make certain even though it is pretty easy

9. matrix multiplication

a. Definition: Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix. Then the product AB is the $m \times p$ matrix C whose entry c_{ij} is formed by: Summing the products obtained by multiplying, in order, each entry in row i of A by the corresponding entry in column j of B.

b. Examples

1) $$\underset{\substack{(2\times 3)\\ A}}{\begin{bmatrix} 1 & 4 & 1 \\ 2 & 1 & -1 \end{bmatrix}} \underset{\substack{(3\times 2)\\ B}}{\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}} = \begin{bmatrix} 1+4+1 & 2+12+1 \\ 2+1-1 & 4+3-1 \end{bmatrix} = \underset{C}{\begin{bmatrix} 6 & 15 \\ 2 & 6 \end{bmatrix}}$$

2) $$\underset{B}{\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 4 & 1 \\ 2 & 1 & -1 \end{bmatrix}}$$ not conformable for multiplication

3) For multiplication to be defined; i.e. for $AB = C$, then number of columns in A must equal the number of rows in B. In above examples, AB is defined but BA not defined

4) $$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix} = 0 + 12 + 3 = 15$$

5) $$\begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 6 & 12 & 18 \\ 1 & 2 & 3 \end{bmatrix}$$

6) $IA = AI = A$ if both are conformable for multiplication

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix}$$

c. Homework

pp 273 - 275 # 7, 19, 25, 27, 37, 45, 69

10. Matrix Inversion

a. Let A be a square matrix; i.e., a matrix with an equal number of rows and columns. If there exists a matrix C such that $CA = AC = I$ then C is called the ~~inv~~ inverse of A and is denoted as A^{-1}

b. Example, let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \quad ; \quad A^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

c. If A^{-1} exists, then it is unique

d. Rule for finding the inverse of a 2×2 matrix (if it exists)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \frac{1}{ad - cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad A^{-1} = \frac{1}{1-4}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

11. Reconsider the two equations

$p = 15 - q$ "demand"

$p = 3 + 2q$ "supply"

Can solve for equilibrium values of p and q using matrix algebra. To do this, it is helpful to write the two equations as

$$p + q = 15$$
$$p - 2q = 3$$

Or, as

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 15 \\ 3 \end{bmatrix}$$

This puts the system in the form $Ax = B$

To solve, find A^{-1}

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \quad \text{So } A^{-1} = \frac{1}{-2-1} \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

verify

$$AA^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So, the solutions would be

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 15 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 + 1 \\ 5 - 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \end{bmatrix}$$

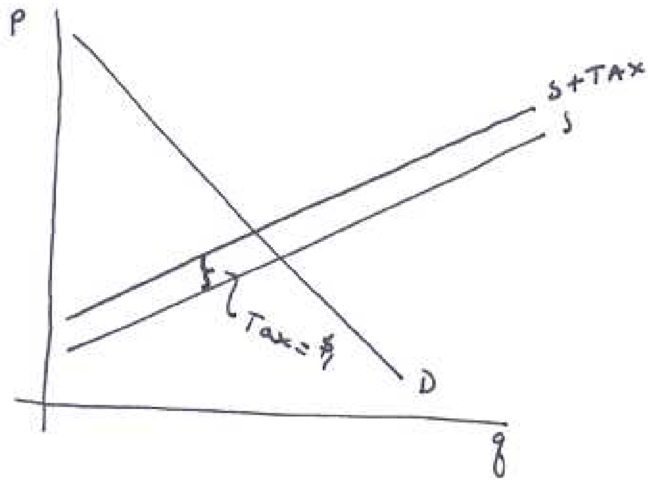
which is the same solutions as before

12. Suppose we put an ~~excise~~ excise tax of \$1 on in this market

Then,

$p = 15 - q$ "demand"

$p = 4 + 2q$ "supply"



Suppose we want to solve for the new equilibrium p and q

$$p + q = 15$$
$$p - 2q = 4$$

So

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 15 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 15 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 + 4/3 \\ 5 - 4/3 \end{bmatrix} = \begin{bmatrix} 11.33 \\ 3.67 \end{bmatrix}$$

Summary

Before tax total revenue $(pq) = 44.00$

After tax total revenue $= 41.58$

Tax collections are \$3.67

Consumers pay $.33\,(3.67) = 1.21$

Producers pay $.67\,(3.67) = \underline{2.46}$

\$3.67

13. Another example: Suppose

$$p = 20 - 2q \qquad \text{"demand"}$$
$$p = 4 + 3q \qquad \text{"supply"}$$

So
$$p + 2q = 20$$
$$p - 3q = 4$$

$$\begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 20 \\ 4 \end{bmatrix}$$

If $A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}$ $\qquad A^{-1} = \dfrac{1}{-3-2} \begin{bmatrix} -3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3/5 & 2/5 \\ 1/5 & -1/5 \end{bmatrix}$

Verify $\begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 3/5 & 2/5 \\ 1/5 & -1/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Therefore
$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 3/5 & 2/5 \\ 1/5 & -1/5 \end{bmatrix} \begin{bmatrix} 20 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 + 8/5 \\ 4 - 4/5 \end{bmatrix} = \begin{bmatrix} 13\tfrac{3}{5} \\ 3\tfrac{1}{5} \end{bmatrix}$$