# Chapter 10
# Limits

1. Definition: The limit of $f(x)$ as $x$ approaches the value "a" is the number $L$, written

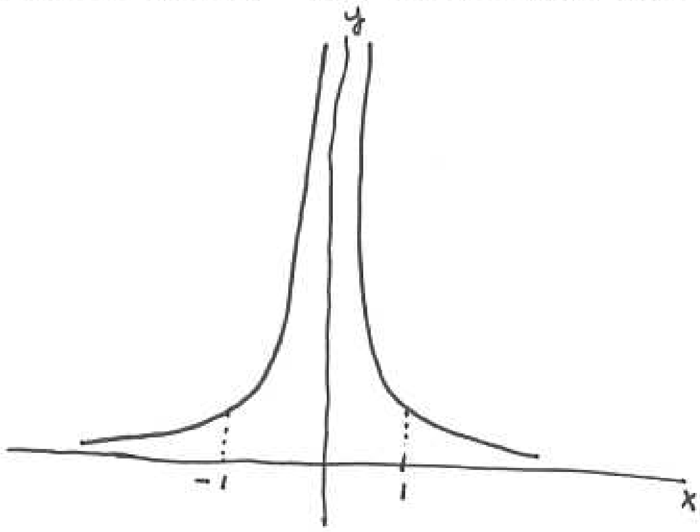
$$\lim_{x \to a} f(x) = L$$

provided that $f(x)$ is arbitrarily close to $L$ for all $x$ sufficiently close to, but not equal to, $a$.

Example #1 $\lim_{x \to 2} (x+3) = 5$

Example #2: Find $\lim_{x \to 0} (1/x^2)$

As $x \to 0$, $f(x) = 1/x^2$ is becoming large (infinite) and do not approach a finite number ($L$). This can be seen from the graph of this function




Hence, the limit does not exist.

2. Properties of limits

a. If $f(x) = c$ then $\lim_{x \to a} f(x) = \lim_{x \to a} c = c$

b. $\lim_{x \to a} x^n = a^n$ for any positive integer $n$.

Examples:

$$\lim_{x \to 2} 7 = 7$$

$$\lim_{x \to 6} x^2 = 6^2 = 36$$

$$\lim_{t \to -2} t^4 = (-2)^4 = 16$$

c. ~~then~~ Assume that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then

$$\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

d. $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

e. $\lim_{x \to a} [c f(x)] = c \cdot \lim_{x \to a} f(x)$

Examples

$$\lim_{x \to 2} (x^2 + x) = \lim_{x \to 2} x^2 + \lim_{x \to 2} x = 4 + 2 = 6$$

$$\lim_{g \to -1} (g^3 - g + 1) = (-1)^3 - (-1) + 1 = -1 + 1 + 1 = 1$$

$$\lim_{x \to 2} (x+1)(x-3) = \lim_{x \to 2} (x+1) \cdot \lim_{x \to 2} (x-3) = 3(-1) = -3$$

Note also: $\lim_{x \to 2} (x+1)(x-3) = \lim_{x \to 2} [x^2 - 2x - 3] = -3$

$$\lim_{x \to -2} 3x^3 = 3(-2)^3 = -24$$

Let $f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$

$$\lim_{x \to a} f(x) = c_n a^n + c_{n-1} a^{n-1} + \cdots + c_1 a + c_0 = f(a)$$

Thus if $f(x)$ is a polynomial function, then $\lim_{x \to a} f(x) = f(a)$

f. If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \qquad \text{provided } \lim_{x \to a} g(x) \neq 0$$

g. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ if $n$ is even then $f(x)$ must be $\geq 0$.

Examples:

$$\lim_{x \to 1} \frac{2x^2 + x - 3}{x^3 + 4} = \frac{2 + 1 - 3}{1 + 4} = \frac{0}{5} = 0$$

$$\lim_{x \to 3} \sqrt[3]{x^2 + 7} = \sqrt[3]{16} = \sqrt[3]{8 \cdot 2} = 2\sqrt[3]{2}$$

$$\lim_{t \to 4} \sqrt{t^2 + 1} = \sqrt{17}$$

3. Thus far we have considered some easy cases where a limit can be found simply by substituting. What if this approach is not possible?

Find $\lim_{x \to -1} \frac{x^2 - 1}{x + 1}$

As $x \to -1$ denominator $\to 0$ so cannot use f (quotients) above. However,

$$\lim_{x \to -1} \frac{x^2 - 1}{x + 1} = \lim_{x \to -1} \frac{(x+1)(x-1)}{x + 1} = \lim_{x \to -1} x - 1 = -2$$

Another example

$$\text{Find } \lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{x-1} = \lim_{x\to 1} (x^2+x+1) = 3$$

A more general and more important example

Let $f(x) = x^2+1$. Find $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$\frac{f(x+h)-f(x)}{h} = \frac{[(x+h)^2+1]-[x^2+1]}{h}$$

So

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h\to 0} \frac{2xh+h^2}{h} = 2x$$

Homework: p.506 # 9, 11, 13, 15, 21, 23, 35, 37, 41

4. A function $f$ is <u>continuous</u> at $a$ if and only if the following three conditions are met

   a. $f(a)$ is defined at $x=a$; that is $a$ is in the domain of $f$

   b. $\lim_{x\to a} f(x)$ exists

   3. $\lim_{x\to a} f(x) = f(a)$

From a practical viewpoint, this means that a continuous function does not have any jumps or breaks. Our work for the remainder of the course will look only at continuous functions, maybe with a rare exception

5. Let's take a closer look at

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

a. In taking this limit we are considering two points in the domain of f. These are x and x+h. These points are h units apart.

b. f(x+h) and f(x) are the "y-values" of ~~the~~ associated with the two values of x

c. Thus,

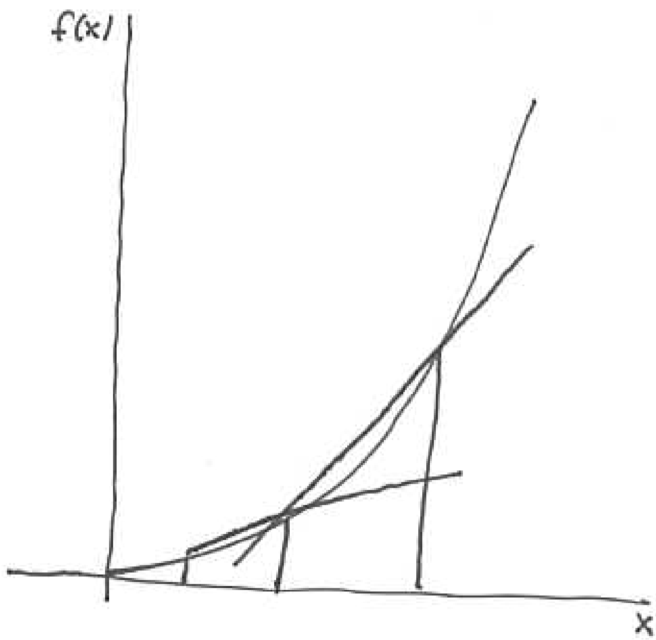
$$\frac{f(x+h) - f(x)}{h} = \frac{f(x+h) - f(x)}{(x+h) - x} \approx \frac{y_2 - y_1}{x_2 - x_1}$$

Can be interpreted as the slope of the function

1) If the function is linear then it does not matter which two points are ~~chosen~~ chosen to evaluate the slope.

2) If the function is not linear, then it does matter. Consider:

$$f(x) = x^2$$




The idea behind the diagram is to show that the slope of $f(x) = x^2$ depends on the points (values ~~f~~ of x) chosen to make the evaluation. And, if two ~~points~~ x-points are chosen, we really get the average of the slopes between these points.

Diagram of function in the first quadrant <u>only</u>.

3) Now, reconsider

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This limit is aimed at finding the slope of a function as the two x-points become closer and closer together (i.e., as $h \to 0$)

So as $h \to 0$, it gives the slope of $f(x)$ at a particular x-point.

This is called the <u>derivative</u> of $f(x)$ and is the subject of Chapters 11-13.

6. Infinite limits

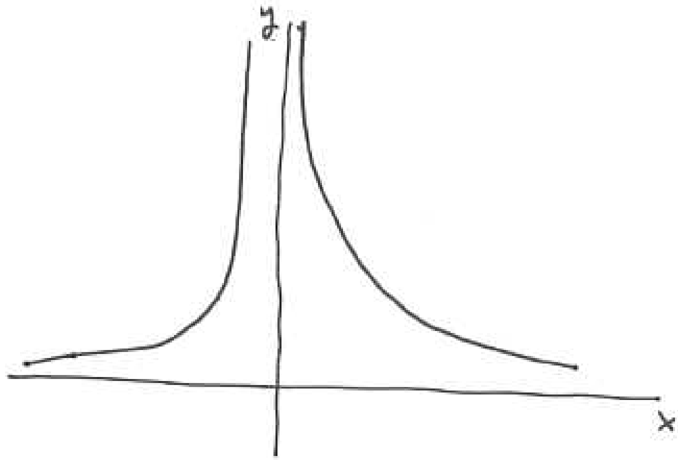
a) Consider

$$\lim_{x \to 0} \frac{1}{x^2}$$

So, as $x \to 0$, $\frac{1}{x^2} \longrightarrow \infty$

Thus

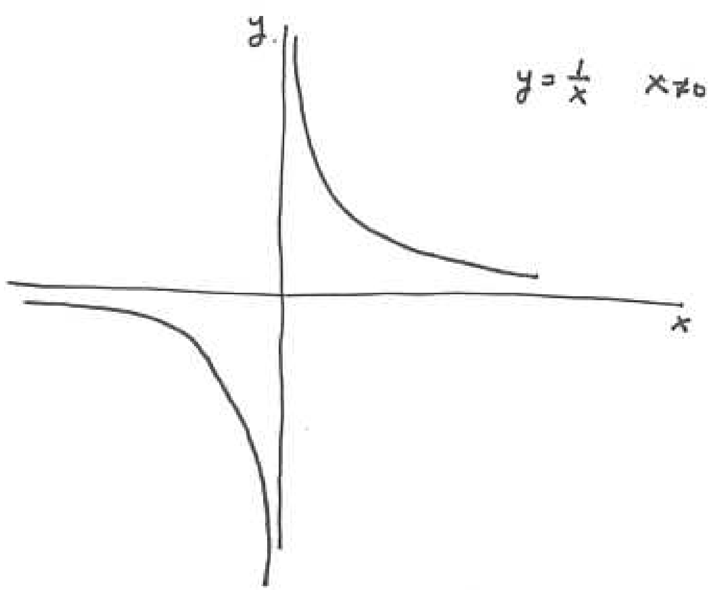
$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$




Notice that as $x \to 0$ from either the right or the left, $f(x) \to \infty$

b) Next look at

$$\lim_{x \to 0} \frac{1}{x}$$




As $x \to 0$ from the left, $\frac{1}{x} \to -\infty$

As $x \to 0$ from the right $\frac{1}{x} \to \infty$

But $\lim_{x \to 0} \frac{1}{x}$ does not exist

c. $\lim_{x \to 2} \frac{x+2}{x^2-4} = \lim_{x \to 2} \frac{x+2}{(x+2)(x-2)} = \lim_{x \to 2} \frac{1}{x-2}$

As $x \to 2$ from the left (i.e. from below 2), then

$$\lim_{x \to 2} \frac{1}{x-2} = -\infty$$

As $x \to 2$ from above 2 (from the right), then

$$\lim_{x \to 2} \frac{1}{x-2} = \infty$$

Hence, ~~the limit~~ of $\lim_{x \to 2} \frac{x+2}{x^2-4}$ is neither $+\infty$ nor $-\infty$. Limit does not exist.

7. Limits at infinity

a. $\lim_{x \to \infty} \frac{4}{(x-5)^3} = 0$

b. $\lim_{x \to -\infty} \sqrt{4-x} = \infty$

c. $\lim_{x \to \infty} \frac{4x^2+5}{2x^2+1} = \dfrac{\lim_{x \to \infty} \left[\frac{4x^2}{x^2} + \frac{5}{x^2}\right]}{\lim_{x \to \infty} \left[\frac{2x^2}{x^2} + \frac{1}{x^2}\right]} = \frac{4}{2} = 2$

d. $\lim_{x \to \infty} \frac{x^2-1}{7-2x+8x^2} = \dfrac{\lim_{x \to \infty} \left[\frac{x^2}{x^2} - \frac{1}{x^2}\right]}{\lim_{x \to \infty} \left[\frac{7}{x^2} - \frac{2x}{x^2} + \frac{8x^2}{x^2}\right]} = \frac{1}{8}$

e. $\lim_{x \to \infty} \frac{x^5-x^4}{x^4-x^3+82} = \infty$ because the largest power of $x$ in the numerator exceeds that in the denominator

Homework: pp 515–516, #9, 21, 31, 37. Then do a few more.

# Requested Example

Suppose that Ms. A. buys a zero-coupon bond maturing in 30 years. The current interest rate is 15%. The face-value of the bond is \$10,000. How much does the bond cost today? Compounding is annual.

$$V = \frac{\$10{,}000}{(1.15)^{30}} = \frac{\$10{,}000}{66.21} = \$151.03 \longleftarrow$$

This is the amount that when compounded at 15% becomes \$10,000 after 30 years.

Suppose that Ms. A. holds the bond for four years. Over this time, assume that ~~the~~ interest rates fall from 15% to 6%. What is the value of the bond after 4 years

$$V = \frac{\$10{,}000}{(1.06)^{26}} = \$2198.10 \longleftarrow$$

This is the amount that when compounded at 6% becomes \$10,000 after 26 years.

If she sells the bond, what rate of return did she earn?

$$\$2198.10 = (1+r)^4 \, 151.03$$

So $$(1+r)^4 = \frac{2198.10}{151.03} = 14.55$$

OR $$(1+r) = (14.55)^{1/4} = 1.95$$

So $r = 0.95$ ; Ms A. made 95% per year by holding the zero-coupon bond for four years.

What if interest rates had risen over this period from 15% to 18%. Then the value after 4 years would be

$$V = \frac{\$10{,}000}{(1.18)^{26}} = \$135.23$$

How about if interest rates had stayed the same: $V = \frac{10{,}000}{(1.15)^{26}} = \$264.5$